Recap VL

Montag, 29. Mai 2017 15:25

Variational Formulation
by integrating PDE on both sides and then integrating by parts
providing we chose the right function space.

Finite Elements formulations
We are usually not generating a mesh ourselves.
Have triangles similarly sized.

h = maximum diam (K)
choice of finite demensional subspace V_h
Very important to choose a suitable bassis for v_h -> hat functions
$V_h = \{v \in V, v\ continuous, v\ on\ k\ is\ linear\}$
$v_h \subset H_0^1()$

Reduction to a linear system
Stiffness matrix is not neccessarily symmetric

To learn
Given an elliptic timeindependent pde:

- coercivity, continuous, bilinear form, linear form, variational formulation, and so
- how to choose function space (H1, H0, H2, H1_0, …)
- basis like hat function. what would be a basis for piecewise quadratic
- how to compute the load vector, stiffness matrix
- error analysis, Galerkin orthogonality

Error analysis:
see photos.
Galerkin orthog. + continuity + coercivity =>
$|e_n|_v \leq C||u - w||_V, \forall w \in V_n$
example on right side of photo. Bounds the FEM error (left side)

it is important to know the regularity. when to use piecewise quadratic finite elements and when not. what is less work if more work would be unneccessary.

Example: Piecewise linear and piecewise squared problems with two convergence graphs. Which one is generated by which?
Then something has the same slope as the other. argue why.
(more intuitive, less deep math).

--------END FEM-----------
--------Begin Finite differences------------

1d Heat equation
u_t = u_xx
with given u(x,0) and boundary u(0,t)=u(1,t)=0
Finite difference method.
'smooth something something diffusion'
Forward euler in time, central in the space

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

$u_j^0 = u_0\left(x_j\right)$

$u_0^n = u_{N+1}^n = 0$
This is explicit because we can rewrite it to
$u_j^{n+1} = \lambda u_{j-1}^n + (1-2\lambda)u_j^n + \lambda u_{j+1}^n$
$\lambda = \frac{\Delta t}{2\Delta x^2}$
This is only conditionally stable.
Example question: sine wave in one case heavy oscillation and blowup, in the other case not. why?

Discrete maximum principles:
    if CFL condition:
        $\lambda \leq \frac{1}{2}\ is\ satisfied$
        $\bar{u}^n = \max_j u_j^n$
        $u_j^{n+1} \leq \lambda \bar{u}^n + (1-2\lambda)\bar{u}^n + \lambda \bar{u}'n \leq \bar{u}^n$
        "maximum principle". ubar is maximum of all us.
    $\Delta t < \frac{\Delta x^2}{2}, \Delta t\ has\ to\ be\ small$

How to rectify that? By going implicit.
Backward difference instead of forward
implicit see photo.

maximum principle.
how you choose the cfl (easy to come up with)
based on order of accuracy and stability, choose what version.
We are unable to justify why it works with the maximum principle. we needed the energy principle.

Linear Transport equation
Forward in time, central in space did not work
Things can move from left to the right but not reverse, so it doesn't make sense that taking info from both sides works.
=> upwind scheme
$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a^+\left(\frac{u_j^n - u_{j-1}^n}{\Delta x}\right) + a^-\left(\frac{u_{j+1}^n - u_j^n}{\Delta x}\right) = 0$
a+ if a positive, else a-
last on photo: diffusion of energy
=> this is why we don't use

should learn method of characteristics on how to solve
upwind are what we always use to solve this

"hyperbolic are simpler to compute in a way because their speed of propagation is finite, so we don't really need every point. but for parabolic we do" nonlinear are really harder though.

time-dependent is parabolic, time independed is hyperbolic: this is not really correct but morally.

scalar conservation laws
$u_t + f(u)_x = 0$
$u(x,0) = u_0(x)$
Ex. Burgers' Equation $u_t + \left(\frac{u^2}{2}\right)_x = 0$
Facts: Even if u_0 is smooth, it can be discontinuous (shocks)

No pointwise solutions -> need weak solutions. u ∈ L inf (r × R)
"you know this or at least i expect you to know"

What happens if the derivation becomes infinitely large? (nonlinearity)
Selfsimiliarity: Infinitely many weak solutions. So you need entropy conditions
It would be good to know the solutions of riemann problems. That can be easily asked in the exams.

Self-similar "rarefaction" solutions.

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$f'^{-1}\left(\frac{x}{t}\right)$

Need to approx. entropy solutions => Finite Volume methods
Cell averages $u_j^n \approx \frac{1}{\Delta x}\int u(v,n)dx$ or smthng like this
rate of change is given by fluxes.
It's important to know the context. why not use finite difference here?
(when it's not nicely defined)

FVM

$$\frac{u_j^{n+1}-u_j^n}{\Delta \mathrm{t}}+\frac{F_{j+\frac{1}{2}}^n-F_{j-\frac{1}{2}}^n}{\Delta \mathrm{x}}=0$$

$$F_{j+\frac{1}{2}}^n \approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f\left(u\left(x_{i+\frac{1}{2}},t\right)\right)dt$$

superpositions of riemann problems.

Godunov scheme
at each interface at time t^n
Riemann ...
see photo
Godunov Flux. Remember this.
small formula by mishra for convex or concave functions.

Lax friedrichs
rusanov

they are all monotone, consistent, conservative (mcc) schemes.
For these schemes, we derive the maximum principle and the TVD property.

typical question: add the fluxes of the two. why is it the way it is? Reason: TVD or not.

What I wanted you to learn:
given a pde, what it is (elliptic and so on…), what its characteristics is, what method to use (there are usually packages but which one do we use?). It's important to know when something does work (when converging at rate 1, 2, 1/2) and when not.
Basic things like maximum principle, energy principle.
Play around pen&paper before you try to implement. What do you expect.
"Even though this is a basic course, you more or less have the basics for more advanced… stuff"

Q: Difference between H space and C space.
A: H is about square differentiability of derivative. C about continuity