Notes

Freitag, 30. Juni 2017 10:46

Boundaries
Dirichlet specifies the values at the boundary.
Neumann specifies the derivatives at the boundary.
Periodic boundary conditions say $U_0^n = U_N^n, U_{N+1}^n = U_1^n$
Non-reflecting Neumann artificial boundary conditions: $U_0^n = U_1^n,\ U_{N+1}^n = U_N^n$
Partitioning
$A\,\mu = f \Leftrightarrow \begin{bmatrix} A_{\partial\partial} & A_{\partial 0} \\ A_{0\partial} & A_{00} \end{bmatrix} \begin{bmatrix} \mu_\partial \\ \mu_0 \end{bmatrix} = \begin{bmatrix} f_\partial \\ f_0 \end{bmatrix}$
We already know μ_∂ from the boundary condition, so we partition the matrix
$\Rightarrow A_{0\partial}\mu_\partial + A_{00}\mu_0 = f_0$
$A_{00}\mu_0 = f_0 - A(0\partial)\ \mu_\partial$

Formulae
Gauss Theorem
$\int_\Omega \nabla \cdot \mathrm{F}\ \mathrm{dx} = \int_{\partial\Omega} F \cdot \vec{n}\, dS$
F is a vector

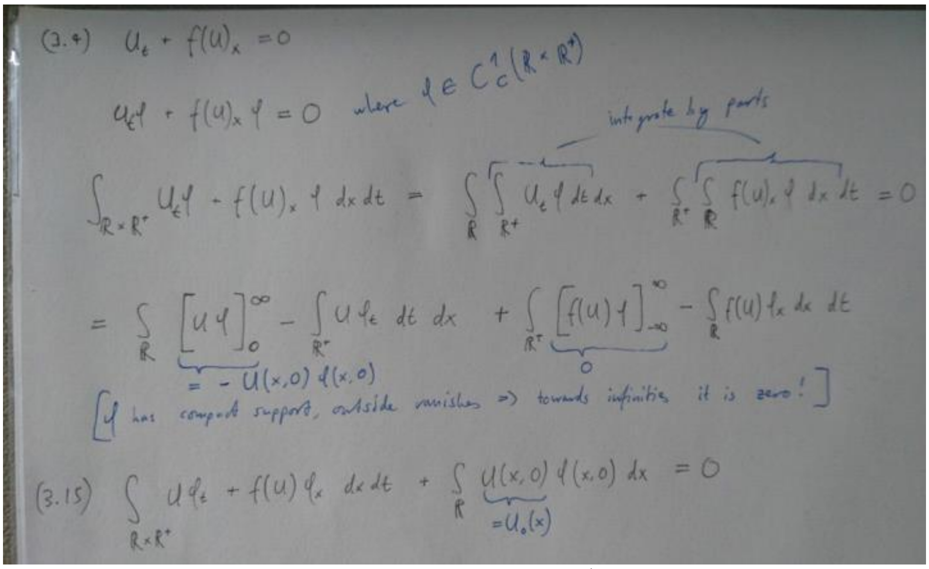
vectorized Product rule
$\nabla \cdot (jv) = (\nabla \cdot j)v + \nabla v \cdot j$
where $\nabla \cdot$ is the divergence and ∇ by itself is the gradient
Divergence is the scalar product of the lying nabla vector and the other vector (added partial products)

and integrated:
$-\int_\Omega \nabla \cdot (\vec{j}v)\, dx = -\int_\Omega (\nabla \cdot \vec{j})v\, dx - \int_\Omega (\nabla v)\vec{j}\, dx$

Green's Formula for R^2
$-\int_\Omega (\nabla \cdot j\ )\ \ v\ \ dx = -\int_{\partial\Omega} j \cdot \vec{n} \cdot v\, dS + \int_\Omega j \cdot \nabla v\ \ dx$
very useful if $v = 0\ on\ \partial\Omega$

other green stuff:
$\int_U \phi\nabla^2 u + \nabla\phi \cdot \nabla u\ \ dU = \int_{\partial U} \phi \frac{\partial u}{\partial n}\, dS\,, \qquad \frac{\partial u}{\partial n} = \nabla u \cdot \vec{n}$
$\int_U u\nabla^2 v\, dU = \int_{\partial U} u \frac{\partial v}{\partial n}\, dS - \int_U \nabla u \cdot \nabla v\ \ dU$

Integration by parts
$\int_\Omega \phi\ div\ \vec{v}\, dV\ = \int_{\partial\Omega} \phi\vec{v} \cdot d\vec{S}\ - \int_\Omega \vec{v} \cdot grad\ \phi\, dV$
$\int_\Omega \frac{\partial u}{\partial x_i} v\ \ d\Omega = \int_{\partial\Omega} u\ v\ \vec{n}_i\ \ d(\partial\Omega) - \int_\Omega \frac{\partial v}{\partial x_i}\, u\, d\Omega \quad //maybe\ incorrect$
For scalar output: $\int_\Omega u_x\ *\ v\, dx = \int_\Omega u * v\ \ dx - \int_\Omega u\ * v_x\ \ dx$



another example: $\int_\Omega uu_{xx}\,dx = -\int_\Omega u_x^2 dx + uu_x\big|_0^1$

Integration by Substitution

$$\int_{\gamma(a)}^{\gamma(b)} f(x)\,dx = \int_a^b f(\gamma(t))\cdot\gamma'(t)\,dt$$

$$\int_{\gamma(U)} f(v)\,dv = \int_U f(\gamma(u))\,|\det(D\,\gamma)(u)|\;du$$

Chain Rule

$$J_{u\circ v}(a) = J_u(v(a))\cdot J_v(a)$$

$$\frac{du}{dr} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

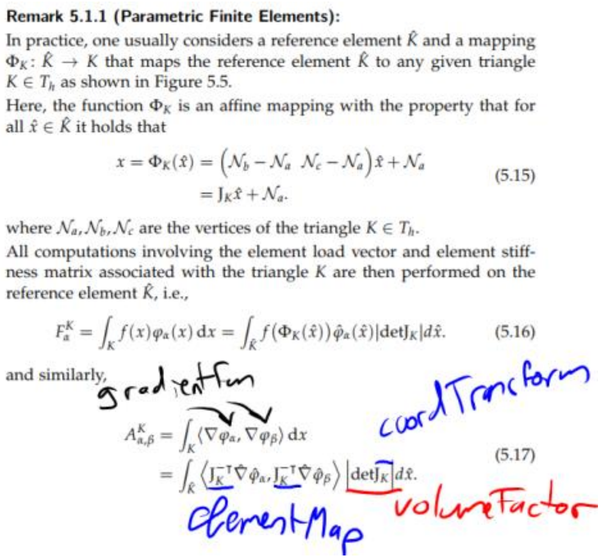
beide richtungen, x und y, beachten!! weil partielle ableitungen

Trick: $\frac{\partial}{\partial x}\frac{\partial u}{\partial \theta} \Rightarrow \frac{\partial^2 u}{\partial \theta^2}\frac{\partial \Theta}{\partial x}$ Ziel: möglichst viele bekannte terme erstellen wie $\frac{\partial \Theta}{\partial x}$ indem man den linken teil zweiteilt.

Triangle Inequality

$$\|x+y\| \le \|x\| + \|y\|$$

Reference Shape


Remark 5.1.1 (Parametric Finite Elements):
In practice, one usually considers a reference element $\hat{K}$ and a mapping $\Phi_K: \hat{K} \to K$ that maps the reference element $\hat{K}$ to any given triangle $K \in \mathcal{T}_h$ as shown in Figure 5.5.
Here, the function Φ_K is an affine mapping with the property that for all $\hat{x} \in \hat{K}$ it holds that

$$x = \Phi_K(\hat{x}) = \begin{pmatrix}\mathcal{N}_b - \mathcal{N}_a & \mathcal{N}_c - \mathcal{N}_a\end{pmatrix}\hat{x} + \mathcal{N}_a = \mathrm{J}_K\hat{x} + \mathcal{N}_a. \quad (5.15)$$

where $\mathcal{N}_a, \mathcal{N}_b, \mathcal{N}_c$ are the vertices of the triangle $K \in \mathcal{T}_h$.
All computations involving the element load vector and element stiffness matrix associated with the triangle K are then performed on the reference element $\hat{K}$, i.e.,

$$F_a^K = \int_K f(x)\varphi_a(x)\,\mathrm{d}x = \int_{\hat{K}} f(\Phi_K(\hat{x}))\hat{\varphi}_a(\hat{x})|\det \mathrm{J}_K|\,d\hat{x}. \quad (5.16)$$

and similarly,

$$A_{\alpha,\beta}^K = \int_K \langle\nabla\varphi_\alpha, \nabla\varphi_\beta\rangle\,\mathrm{d}x = \int_{\hat{K}} \langle \mathrm{J}_K^{-T}\hat{\nabla}\hat{\varphi}_\alpha, \mathrm{J}_K^{-T}\hat{\nabla}\hat{\varphi}_\beta\rangle |\det \mathrm{J}_K|\,d\hat{x}. \quad (5.17)$$

polar coordinates

$x(r,\Theta) = r\cdot\cos(\Theta)$, $\quad y(r,\Theta) = r\cdot\sin(\theta)$, $\quad (r,\Theta) \in [0,\infty[\,\times\,[0,2\pi[$

for a function $u: \Omega \times [0,T] \mapsto \mathbb{R}$, define $\tilde{u}(r,\theta,t) = u(r\cdot\cos(\theta), r\cdot\sin(\theta), t)$.

On the unit disk, r ∈ *[0,1[*

Laplace Operator

Laplace einer funktion f(x,y,t) ist die zweite ableitung in x und in y Richtung. Nicht in t.

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \qquad \Delta \tilde{u} = \frac{\partial^2 \tilde{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{u}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \tilde{u}}{\partial \phi^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{u}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \tilde{u}}{\partial \phi^2}$$

Taylor

$$T_n\big(f(x,a)\big) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Rest $R_n f(x,a) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt$

Gronwall: If the derivative and the function are together in an inequality

Theorem A.1 (Gronwall's inequality). *Let $\beta(t)$ be continuous and $u(t)$ be differentiable on some interval $[a,b]$, and assume that*

$$u'(t) \leqslant \beta(t)u(t) \qquad \forall\, t \in (a,b).$$

Then

$$u(t) \leqslant u(a)\exp\left(\int_a^t \beta(t)\right) \qquad \forall\, t \in [a,b].$$

Transformation

Given a function that takes f and returns the integral of f on [-1,1], insert the argument as $f * 0.5 * (b-a).norm(⬚)$

Equations

Poisson (FDM) *Elliptic time-dependent*

$-u''(x) = f(x)$

Maximum Principle satisfied: $\|u\|_{\Delta x,\infty} \leq \frac{1}{8}\|f\|_{\Delta x,\infty}$, where $\|q\|_{\Delta x,\infty} := \sup_{1\leq j\leq N} \left|q\left(x_j\right)\right|$

Can be solved explicitly in 1D using Greens function (or with finite elements):

$$u(x) = \int_0^1 G(x,y)f(y)\,dy. \qquad G(x,y) = \begin{cases} y(1-x), & 0 \leq y \leq x \\ x(1-y), & x \leq y \leq 1 \end{cases}$$

G here is continuous, symmetric, $\geq 0 \forall x, y \in]0,1[$, for a fixed x or y, it is piecewise linear in the other variable.

Any solution w to the poisson equation with righthandside g(x) satisfies the estimate $\|w\|_\infty \leq \frac{1}{8}\|g\|_\infty$

Poisson in 2D

$-\left(\frac{u_{i+1,j}-2u_{i,j}+u_{i+1,j}}{\Delta x^2} + \frac{u_{i,j+1}-2u_{i,j}+u_{i,j+1}}{\Delta y^2}\right) = f_{i,j},\ F = \Delta x^2 * \left[f_{00}, \ldots, f_{N-1,N-1}\right]$, A ist tridiagonal mit B auf Hauptdiag. und $(-I)$ auf nebendiag. B ist 4 auf Hauptdiag und -1 auf nebendiag. $A\mu = F$

Media Porous (FDM)

$-\nabla \cdot (\sigma \nabla u) = f\ \ in\ \Omega := (0,1)^2$

$\Rightarrow f = -\frac{\partial}{\partial x}\left(\sigma\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(\sigma\frac{\partial u}{\partial y}\right)$ *where σ is a function*

$$\frac{\partial}{\partial x}\left(\sigma\frac{\partial u}{\partial x}\right) = \frac{\sigma\left(x_{i+\frac{1}{2}}, y_j\right)}{h^2}u_{i+1,j} - \frac{\sigma\left(x_{i+\frac{1}{2}}, y_j\right) + \sigma\left(x_{i-\frac{1}{2}}, y_j\right)}{h^2}u_{i,j} + \frac{\sigma\left(x_{i-\frac{1}{2}}, y_j\right)}{h^2}u_{i-1,j}$$

which gives if also done for the right side:

$$-f_{i,j} = -\frac{\sigma_{i+\frac{1}{2},j}}{h^2}u_{i+1,j} - \frac{\sigma_{i-\frac{1}{2},j}}{h^2}u_{i-1,j} - \frac{\sigma_{i,j+\frac{1}{2}}}{h^2}u_{i,j+1} - \frac{\sigma_{i,j-\frac{1}{2}}}{h^2}u_{i,j-1} + \frac{\sigma_{i+\frac{1}{2},j} + \sigma_{i-\frac{1}{2},j} + \sigma_{i,j+\frac{1}{2}} + \sigma_{i,j-\frac{1}{2}}}{h^2}u_{i,j}$$

Heat Equation (FDM) *Parabolic linear*

$$\frac{\partial u}{\partial t} - \Delta u = 0 \quad on\ \Omega \times [0,T]$$

$$u\Big|_{\partial\Omega} = 0$$

$$u(x,y,0) = u_0(x,y)$$

Total thermal **Energy** that must decrease over time: $\int_\Omega u(x,t)\ dx$ is originally $\boldsymbol{\varepsilon(t) = \frac{1}{2}\int_0^1 |u(x,t)|^2\ dx}$
Boundary Condition $-\nabla u \cdot n = \gamma u \quad on\ \partial\Omega \times [0,T]\ with\ \gamma > 0$ stands for convective cooling.
Assume Temperature to never be negative.
Maximum Principle: $\min\left(0, \min_\chi\left(u_0(\chi)\right)\right) \leq \mathrm{u(x,t)} \leq \max\left(0, \max_\chi\left(u_0(\chi)\right)\right)$
More information on stability at FTCS

Linear Transport equation in 1D (FDM) *Hyperbolic linear*
$u_t + c \cdot u_x = 0$

$$\frac{\partial u}{\partial t}(x,t) + a(x)\frac{\partial u}{\partial x}(x,t) = 0, \qquad (x,t) \in (0,1) \times \mathbb{R}$$
$$u(0,t) = u(1,t), \qquad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t), \qquad t \in \mathbb{R}$$
$$u(x,0) = u_0(x)$$

To solve with method of characteristics
//upwind scheme usable here
Radiative Cooling: Energy decreases with time => $\frac{dE(t)}{dt} = \frac{d\int_\Omega u(x,t)dx}{dt} < 0$
$$u_t - \Delta u = 0, \qquad -\nabla u \cdot n = \gamma u\ \ on\ \partial\Omega, \qquad u(x,0) = u_0(x)\ on\ \Omega$$

For hyperbolic advection(transport) equations, any choice of timestep results in an unstable FTCS scheme.

Burger's Equation (FVM) *Hyperbolic*
inviscid Burgers equation: $u_t + uu_x = u_t + \left(\frac{1}{2}u^2\right)_x = 0$
Is a scalar conservation law => fulfills TVD that $\left||u(x,t)|\right| \leq \left||u_0(t)|\right|$
//insert s as condition for x smaller than / bigger than in the weak solution U.

with Method of Characteristics:
$\frac{\partial x}{\partial t} = u, \frac{\partial t}{\partial t} = 1, \frac{du}{dt} = 0\ (const\ auf\ charakteristik)$
$\Rightarrow x'(t) = u(x,t) = u_0(x) \Rightarrow x(t) = t * u_0(x_0) + x_0$ for starting points x_0
//characteristic line goes straight upwards if it contains no t - e.g. if $u_0 = 0$.

Scalar conservation laws (FVM) *Hyperbolic non-linear*
SCLs include Transport equation and Burgers equation. $U_t + a(x,t)U_x = 0$
SCLs satisfy the minimum/maximum principles => stay in interval of start usually
CFL: $\max f'(U_j^n)\frac{\Delta t}{\Delta x} \leq \frac{1}{2}$
Even if u_0 is smooth, the solution can become discontinuous
Need weak solutions
Lemma about **Energy Bound** for Scalar Conservation Laws*:*

Lemma 2.1. *Let $U(x,t)$ be a smooth solution of (2.1) which decays to zero at infinity, i.e, $\lim_{|x|\to\infty} U(x,t) = 0$ for all $t \in \mathbb{R}_+$, and assume that $a \in C^1(\mathbb{R},\mathbb{R}_+)$. Then U satisfies the energy bound*

$$\int_{\mathbb{R}} U^2(x,t)dx \leqslant e^{\|a\|_{C^1}t}\int_{\mathbb{R}} U_0^2(x)dx \tag{2.6}$$

for all times $t > 0$.

TVD
Hyperbolic Equations like the transport equation are TVD if the total variation at step n+1 is not larger than at step n.
$$TV = \int \left|\frac{\partial u}{\partial x}\right| dx = \sum_j \left|u_{j+1} - u_j\right|$$
A numerical scheme is TVD ⇔ it is monotonicity preserving (if u^n is monotonically in/decreasing in space, then so is u^{n+1}.)

Elliptic, Hyperbolic, Parabolic

$$A(x,y)\frac{\partial^2 f}{(\partial x)^2} + B(x,y)\frac{\partial^2 f}{\partial x \partial y} + C(x,y)\frac{\partial^2 f}{(\partial y)^2} = 0$$

$B^2 - 4AC > 0, \quad Hyperbolic, \quad 2\ real\ characteristics$
$B^2 - 4AC = 0, \quad Parabolic, \quad one\ real\ characteristic$
$B^2 - 4AC < 0, \quad Elliptic, \quad no\ real\ Characteristics$
Characteristic is where the equation becomes an ODE

Finite Differences
Convergence of approx order 2
u: value at Point, f: load function

$$f'(x) = \frac{f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)}{h}$$

$$f''(x) = \frac{f'\left(x+\frac{h}{2}\right) - f'\left(x-\frac{h}{2}\right)}{h} = \frac{f(x-h) + f(x+h) - 2f(x)}{h^2}$$

Expressing the given discretization problem as $Au = F$, the **stiffness Matrix** is

$$A := \frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & \ddots & \ddots \\ 0 & -1 & 2 & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & -1 \\ 0 & \ddots & \ddots & -1 & 2 \end{bmatrix}$$

$$u := \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \quad F := \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{pmatrix}$$

the boundary points are given by the **Dirichlet Boundary** and are <u>not</u> contained in u for this excercise with N+1 subintervals. (We can also move the factor $\frac{1}{h^2}$ to the rhs and get $h^2 * F$ which is faster to compute.)

Linear FEM in 1D
Trial function u is piecewise linear on an equidistant mesh => using hat basis to express it

$$u(x) = \sum_{i=0}^{N-1} \mu_i\, \varphi_i(x), \qquad constant\ factor\ \mu\ to\ scale\ the\ basis\ \varphi$$

So starting from a variational formulation $\int_a^b u'(x)v'(x)\,dx = \int_a^b f(x)v(x)\,dx,\ u \in H_0^1([a,b]), \forall v \in H_0^1([a,b])$, we can choose the hat basis function as testfunction v and get by inserting $\int_a^b u'(x)v'(x)\,dx = \sum_{i=0}^{N-1} \mu_i \int_a^b \varphi_i'(x)\varphi_j(x)\,dx \quad \forall j \in [0, N-1]$.
Thus, $A_{j,i} = \int_a^b \varphi_i'(x)\varphi_j'(x)\ dx,\ \mu = \{\mu_i\}_{i=0}^{N-1},\ F_j = \int_a^b f(x)v(x)dx = \int_a^b f(x)\varphi_j(x)\,dx$
for $A\vec{\mu} = \vec{F}$

<u>Variational Formulation</u>
After multiplying both sides with a testfunction $v \in H_0^1(\Omega)$, integrate. Then probably **use Green / partial Integration** (especially for poisson) to get an integral on the boundary, which is zero.
This gives the formulation "Find $u \in V = \{w \in H^1(\Omega): w = g\ on\ \partial\Omega\}$ such that .... " followed by the newly found integral equation.
Here, g is the boundary function from the condition $u(x) = g(x)$.

More general PDE for bilinear form
$-\nabla \cdot (c\nabla u + \alpha u - \gamma) + \beta \nabla u + du - f = 0$ on Ω
boundary conditions: $u = g\ on\ \partial\Omega$
these coefficients can depend on x and y, but not on u
c can even be matrixvalued. α, γ, β *are vectorvalued*
$d, f \in R$

$$\int_\Omega -(\nabla \cdot (c\nabla u + \alpha u - \gamma))v + (\beta \cdot \nabla u + df)v \, dx = 0 \quad \forall v, v = 0 \; on \; \partial\Omega$$

partial integration **and then v=0 on the boundary because it is in H_0^1**

$$\Rightarrow \int_\Omega (c\nabla u + \alpha u - \gamma) \cdot \nabla v + (\beta \cdot \nabla u + df)v \, dx = 0$$

$$\Rightarrow \int_\Omega (c\nabla u) \cdot \nabla v + u(\alpha \cdot \nabla v) + v(\beta \cdot \nabla u) + duv \; dx = \int_\Omega \gamma \cdot \nabla v + fv \; dx$$

choose basis functions and test for all that fulfill the boundary conditions.

Choose $c, \alpha, \beta, \gamma, d, f$ in a nice way to get the PDE of the Core Problems

Quadratic FEM

Error for a quadratic solution is machine precision. We only have an advantage of convergence order in comparison with LinFem if the solution is in H^2.

(EMP: Falls f in H1. Und u in H1 denn linfem order 2 für L2 norm, 1 in H1. Und falls u in H2 denn quadfem order 3, 2 in H1 norm. Das stimmt.)

Exercise 1 Quadratic Finite Elements for the Poisson equation in 2D

We consider the problem

$$-\Delta u = f(\boldsymbol{x}) \quad \text{in } \Omega \subset \mathbb{R}^2 \tag{1}$$

$$u(\boldsymbol{x}) = 0 \quad \text{on } \partial\Omega \tag{2}$$

where $f \in L^2(\Omega)$.

We know that its variational formulation is given by: Find $u \in V = H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) \, d\boldsymbol{x}, \quad \text{for all } v \in H_0^1(\Omega). \tag{3}$$

We solve (3) by means of *quadratic finite elements* on triangular meshes $\mathcal{M}$ of Ω. Consequently, we consider the following finite-dimensional subspace of $H_0^1(\Omega)$:

$$V^h = \{w \colon \Omega \to \mathbb{R} \colon w \text{ is continuous}, w = 0 \text{ on } \partial\Omega, \text{ and } w|_K \text{ is a } \textbf{second order polynomial} \; \forall K \in \mathcal{M}\}.$$

So we consider two types of basis functions. The ones associated to the vertices of the mesh and the ones to the midpoint of each edge i of the mesh.

$$b_i(x_j) = \begin{cases} 1, & i = j \\ 0, & else \end{cases}, i = 0, \dots, (N_{Vertices} - 1)$$

$$\psi_i(m_j) = \begin{cases} 1, & i = j \\ 0, & else \end{cases}, i = 0, \dots, (N_{Edges} - 1)$$

$$\Rightarrow u_N(x) = \sum_{i=0}^{N_V - 1} \mu_i b_i(x) + \sum_{i=0}^{N_E - 1} \mu_{i+N_v} \psi_i(x) = \sum_{i=0}^{N-1} \mu_i \varphi_i(x)$$

Solution: We have

$$\mathbf{A}_{ij} = \begin{cases} \int_\Omega \nabla b_i^N(\boldsymbol{x}) \cdot \nabla b_j^N(\boldsymbol{x}) \, d\boldsymbol{x} & i, j = 0, \dots, N_V - 1, \\ \int_\Omega \nabla b_i^N(\boldsymbol{x}) \cdot \nabla \psi_{j-N_V}^N(\boldsymbol{x}) \, d\boldsymbol{x} & i = 0, \dots, N_V - 1, \; j = N_V, \dots, N - 1, \\ \int_\Omega \nabla \psi_{i-N_V}^N(\boldsymbol{x}) \cdot \nabla b_j^N(\boldsymbol{x}) \, d\boldsymbol{x} & i = N_V, \dots, N - 1, \; j = 0, \dots, N_V - 1, \\ \int_\Omega \nabla \psi_{i-N_V}^N(\boldsymbol{x}) \cdot \nabla \psi_{j-N_V}^N(\boldsymbol{x}) \, d\boldsymbol{x} & i, j = N_V, \dots, N - 1. \end{cases}$$

$$\mathbf{F}_i = \begin{cases} \int_\Omega f(\boldsymbol{x}) b_i^N(\boldsymbol{x}) \, d\boldsymbol{x} & i = 0, \dots, N_V - 1, \\ \int_\Omega f(\boldsymbol{x}) \psi_{i-N_V}^N(\boldsymbol{x}) \, d\boldsymbol{x} & i = N_V, \dots, N - 1. \end{cases}$$

Should give a 6 × 6 Element-Stiffness-Matrix

FEM Error analysis:

$e_n = u - u_n$

Galerkin orthogonality: $a(e_n, w) = 0, \forall w \in V_n$

Galerkin orthog. + continuity + coercivity =>

$|e_n|_V \leq C||u - w||_V \; , \forall w \in V_n$

FEM error ≤ interpolation error

Example: for piecewise linear finite elements with continuity and coercivity

$$||e_n||_{H_0^1} \leq Ch||u||_{H^2(a)}$$

$$||e_n||_{L^2} \leq Ch^2||u||_{H^2(a)}$$

Let V be a real Hilbert space with the norm $\|\cdot\|$. Let $a : V \times V \to \mathbb{R}$ be a bilinear form with the properties

- $|a(v,w)| \leq \gamma\|v\|\,\|w\|$ for some constant $\gamma > 0$ and all v, w in V (continuity)
- $a(v,v) \geq \alpha\|v\|^2$ for some constant $\alpha > 0$ and all v in V (coercivity or V-ellipticity).

Let $L : V \to \mathbb{R}$ be a bounded linear operator. Consider the problem of finding an element u in V such that

$a(u,v) = L(v)$ for all v in V.

Consider the same problem on a finite-dimensional subspace V_h of V, so, u_h in V_h satisfies

$a(u_h,v) = L(v)$ for all v in V_h.

By the Lax–Milgram theorem, each of these problems has exactly one solution. **Céa's lemma** states that

$\|u - u_h\| \leq \frac{\gamma}{\alpha}\|u - v\|$ for all v in V_h.

That is to say, the subspace solution u_h is "the best" approximation of u in V_h, up to the constant γ/α.

The proof is straightforward

$\alpha\|u-u_h\|^2 \leq a(u-u_h, u-u_h) = a(u-u_h, u-v) + a(u-u_h, v-u_h) = a(u-u_h, u-v) \leq \gamma\|u-u_h\|\|u-v\|$ for all v in V_h.

We used the a-orthogonality of $u - u_h$ and V_h

$a(u-u_h, v) = 0, \ \forall\, v$ in V_h

which follows directly from $V_h \subset V$

$a(u,v) = L(v) = a(u_h,v)$ for all v in V_h.

FEM example: expected Convergence of $-\Delta u(x) + cu(x) = f(x)$

linFEM order 2 and 1 for L^2 and H^1 Norms. quadratic FEM orders 3 and 2 respectively. At least if exact u is smooth

LinFEM: The error satisfies the estimate $||u - uh||_{H_0^1}(\Omega) = O(h)$, where u is the exact solution to (1.1)-(1.2) and uh is its linear finite element approximation on a mesh with meshwidth h. Thus, the right error curves are the one with slope 1 with respect to the meshwidth and the one with slope 0.5 with respect to the number of degrees of freedom (because in two dimensions $h = O\left(N^{-\frac{1}{2}}\right)$

Crank-Nicolson Scheme

$\left(\frac{U_j^{n+1}-U_j^n}{\Delta t}\right) = \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{2\Delta x^2} - \frac{U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}}{2\Delta x^2}$ has $AU^{n+1} = F^n$

where $F_j^n = \frac{\lambda}{2}u_{j-1}^n + (1-\lambda)u_j^n + \frac{\lambda}{2}u_{j+1}^n)$ and A is on the main diag. $(1+\lambda)$ and on the second diags $-\frac{\lambda}{2}$

Shape Functions

```
inline double lambda(int i, double x, double y) {
    if (i == 0) {
        return 1 - x - y;
    } else if (i == 1) {
        return x;
    } else {
        return y;
    }
}
```

onedimensional

$\varphi_0^K(x) = (2\lambda_0(x) - 1)\lambda_0(x),$

$\varphi_1^K(x) = (2\lambda_1(x) - 1)\lambda_1(x),$

$\varphi_2^K(x) = (2\lambda_2(x) - 1)\lambda_2(x),$

$\varphi_3^K(x) = 4\lambda_0(x)\lambda_1(x),$

$\varphi_4^K(x) = 4\lambda_1(x)\lambda_2(x),$

$\varphi_5^K(x) = 4\lambda_0(x)\lambda_2(x),$

quadratic

Hat Function Derivative and FEM stiffness Matrix

$$\varphi_j(x)' = \begin{cases} \frac{1}{h}, & \text{if } x \in [(j-1)h, jh], \\ -\frac{1}{h}, & \text{if } x \in [jh, (j+1)h], \\ 0, & \text{otherwise.} \end{cases}$$

implicit Finite Difference Scheme

$AU^{n+1} = F^n = U^n$, A has $(1+2\lambda)$ and $-\lambda, -\lambda$ on the diags

Hence, for $i = 1, \ldots, N$ it holds that

$$A_{ij} = (\varphi_i', \varphi_j') = \begin{cases} 0, & \text{if } |i-j| > 1, \\ -\frac{1}{h}, & \text{if } |i-j| = 1 \\ \frac{1}{h^2}\int_{x_{j-1}}^{x_{j+1}} 1\,\mathrm{d}x = \frac{2}{h} & \text{if } |i-j| = 0. \end{cases}$$

Therefore, the matrix A is given by

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{bmatrix}. \qquad (4.43)$$

Thus, up to a scaling factor, the stiffness matrix A in FEM is identical to the matrix A that arises in a finite difference method.

Norms

maximum Norm

$\left|\left|u - u_{Ni}\right|\right|_{\infty} = \max_{1 \le j \le N} \left|u\left(x_j\right) - u_{Nj}\right|$ where u_N is the discretized solution and u is the exact solution.

Convergence Study

$$e_{k+1} \approx C * e_k^p \Rightarrow \log e_{k+1} \approx \log C + p \log e_k \Rightarrow p \approx \frac{\log(e_{k+1}) - \log(e_k)}{\log(e_k) - \log(e_{k-1})}$$

L2-Norm

$\left|\left|f\right|\right|_{L^2}^2 = \int_{\Omega} \langle f(x), f(x) \rangle \, dx$

Alternativ als Vektornorm l2: Wurzel der Summe aller quadrierten Absolutbeträge.

Inner Product: $(u, v) = \int u \cdot v \, dx$

L1-Norm, generell Lp norm

Vektornorm. Summe aller absoluten Beträge.

generell als Vektornorm: $\left|\left|x\right|\right|_p = \sqrt[p]{\sum |x|^p}$

H1-Norm

$\left|\left|u\right|\right|_{H^1}^2 = \left|\left|u'\right|\right|_{L^2}^2 + \left|\left|u\right|\right|_{L^2}^2$

H1 Norm immer grösser als L2 norm

Inner Product: $(u, v)_{H^1} = \int \langle \nabla u, \nabla v \rangle + \langle u, v \rangle \, dx$

H01-Norm

Inner Product: $(u, v) = \int_{\Omega} \langle \nabla v, \nabla w \rangle \, dx$

$\left|\left|u\right|\right|_{H_0^1}^2 = (u, u) = \int \langle \nabla u, \nabla u \rangle \, dx$

Spaces

$H_0^1([a, b])$ is the function space of all continuous functions v such that

$$\int_a^b |v'|^2 dx \le C \;\; and \;\; v(a) = v(b) = 0$$

Inner Product: $(v, w) = \int_{\Omega} \langle \nabla v, \nabla w \rangle \, dx$

The above sobolev space requires the function to be in $L^2(\Omega)$, which means that it is squre-integrable and the scalar product is defined in L^2 as $\int vw \, dx$.

Q: Difference between H space and C space.

A: H is about square integrability of derivative. C about continuity

Eine funktion ist element des sobolev-raums H^1, wenn sie in L^2 ist und die erste Ableitung square-integrable ist. Das

inner product ist $(v, w) = \int_\Omega \langle \nabla v, \nabla w \rangle + \nabla w \; dx$
H^1 ist ein Hilbertraum, so wie auch H_0^1 und L^2. D.h. es hat ein Skalarprodukt und der Raum ist vollständig bzgl der damit induzierten Norm -> jede Cauchy-Folge konvergiert.
Das Skalarprodukt sei linear im zweiten argument und semilinear im ersten. D.h. mit $\lambda \in \mathbb{C}$ gilt $\langle u, \lambda v \rangle = \lambda \langle u, v \rangle$ und $\langle \lambda u, v \rangle = \bar{\lambda} \langle \mathrm{u}, \mathrm{v} \rangle$.

H^S contains L^2 functions whose weak derivatives of order up to s are also L^2. So if something is in H^1, then its derivative is square integrable.

$C_C^1(\Omega)$ is the space of continuously differentiable functions with compact support on Omega. That is, they vanish outside of it. This implies they will vanish on $\partial\Omega$.

FTCS
Example for 1D heat equation. (For transport: $u_i^{n+1} = u_j^n - \frac{a\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n)$)
$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$
$$\Rightarrow \frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\alpha}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$
$$\Rightarrow u_i^{n+1} = u_i^n + \frac{\alpha \Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$
Stable iff $\frac{\alpha \Delta t}{\Delta x^2} \le \frac{1}{2}$
For hyperbolic advection(transport) equations, any choice of timestep results in an unstable FTCS scheme.

Explicit vs Implicit Euler
$u_{k+1} = u_k + \Delta t \cdot f(t_k, y_k)$, where f approximates the derivative
Implicity would instead have $f(t_{k+1}, y_{k+1})$ with the lhs the same.
Explicit equals forward in time. Central in Space see to the left in FTCS.
Backward in space see Upwind.

Upwind Scheme
The central scheme leads to a growth of energy at every time step and is unstable. We need to find schemes that posses a discrete version of the energy estimate.
Simplest: Forward in time (implicit), backward in space (explicit)
if the speed of the characteristic - a - is positive and the direction of information propagation is from left to right, then we use backward in space. (Example with linear transport eq:)
$$\frac{\partial u}{\partial t} + a * \frac{\partial u}{\partial x} = 0 \Rightarrow \frac{U_j^{n+1} - U_j^n}{\Delta t} + a \frac{\left(U_j^n - U_{j-1}^n\right)}{\Delta x} = 0$$
$$\Rightarrow U_j^{n+1} = U_j^n - \frac{a\Delta t}{\Delta x}\left(U_j^n - U_{j-1}^n\right)$$
and with $a < 0$ we use forward in space instead:
$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a \frac{\left(U_{j+1}^n - U_j^n\right)}{\Delta x} = 0$$
This can generally (for both signs) be formulated and then reformed to
$$\frac{\left(U_j^{n+1} - U_j^n\right)}{\Delta t} + a \frac{\left(U_{j+1}^n - U_{j-1}^n\right)}{2\Delta x} = \frac{|a|}{2\Delta x}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$
$// \; with \; |a| = \max\{a, 0\} - \min\{a, 0\}$
The upwind scheme is **stable** if $c = \left|\frac{a\Delta t}{\Delta x}\right| \le 1$. Then it satisfies the energy estimate $E^{n+1} \le E^n$. where energy is defined as $E^n = \frac{1}{2}\Delta x \sum_j \left(U_j^n\right)^2$.
Energy decreases a little but is mostly conserved.
Important: ratio of $\Delta t \; to \; \Delta x$. To make more accurate, keep ratio and refine mesh.

Solution: Since the direction of propagation is to the *right* (i.e. $a(x) \ge 0, \; \forall x \in \mathbb{R}$), the upwinded FDM scheme uses the *left* differences, combined with the Forward Euler scheme:
$$U_j^{n+1} = U_j^n + a(x_j)\frac{\Delta t}{\Delta x}(U_j^n - U_{j-1}^n),$$
with time step size Δt restricted by the CFL condition
$$\Delta t \le \sup_{x \in \mathbb{R}} |a(x)| \Delta x = \Delta x.$$

FDM
FDM:Derivatives are discretized. e.g. $u''(x_j) \approx \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2}$

FDM for Poisson: Equation $-u_{j+1} + 2u_j - u_{j-1} = (\Delta x)^2 f_j$ leads to $Au = F$, where A is tridiagonal with -1/2/-1 and $F = (\Delta x)^2 \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}$

=> rather large System solving involved in calculating the derivatives. But it is tridiagonal, diagonally dominant, positive definit, symmetric and because of the last two invertible. So we can use any discrete linear solver, e.g. LU decomp. Empirical convergence of the FDM scheme is 2.

Method of Characteristics

$x'(t) = a(x(t), t), \qquad x(0) = x_0$

for the given equation $\frac{dU}{dt} + a(x,t)\frac{dU}{dx} = 0, \forall (x,t) \in \mathbb{R} \times \mathbb{R}_+$

$x(t)$ describes where the solution U stays constant.

Case of constant a: $x(t) = x_0 + a * t$

$\Rightarrow U(x,t) = U_0(x_0) = U_0(x - at)$ //charakteristik eq auflösen nach x0 und das in U als argument einsetzen.

$U_L > U_R \Rightarrow Shock \Rightarrow characteristics\ flow\ into\ the\ shock \Rightarrow multiple\ solutions$

$U_L < U_R \Rightarrow Rarefaction \Rightarrow missing\ information \Rightarrow Rankine - Hugoniot\ to\ find\ a\ possible\ weak\ solution$

note that in that case, we could also add steps in between to get multiple shock lines and still have a weak solution by RH.

So we rather create a rarefaction wave which gives a continuous but not neccessarily differentiable solution.

IN BOTH CASES: The wave speed is bounded in abs value by maximum of $|f'(U_l)|$ and $|f'(U_r)|$

Burgers Equation has characteristics of form u*t+x0 because

u is constant on characteristic x and $\frac{\partial x}{\partial t} = u_x$

//but this arguing doesn't seem to work with rarefaction where x/t cosnt

If the solutions for a double-riemann problem have again meeting caracteristica, take the same flux function and set U_L *and* U_R to the *values* to the left and right of the new problem zone.

Steps from Characteristics to solution

replace boundaries depending on x_0 with x by solving equations like $x = x_0 + t$. Also insert that x in u as parameter

Consider the one-dimensional linear transport equation:

$$\frac{\partial}{\partial t}u(x,t) + a(x)\frac{\partial}{\partial x}u(x,t) = 0, \qquad \forall (x,t) \in \mathbb{R} \times \mathbb{R}_+,$$
$$u(x,0) = u_0(x), \qquad \forall x \in \mathbb{R}, \tag{4.1}$$

with coefficient $a(x)$ given by

$$a(x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } 0 \leq x < 1, \\ 1 & \text{if } 1 \leq x. \end{cases} \tag{4.2}$$

(4a) * Write down the equation of characteristics of (4.1) and sketch characteristic curves in the sub-domain $[-1, 2]$ up to time $t = 1$. Use the equation of characteristics to derive an expression for the exact solution of (4.1) in terms of initial data $u_0(x)$.

HINT: For the characteristics equation, do not forget to specify the initial conditions.

HINT: The solution to ODE of the form $\frac{\partial x(t)}{\partial t} = \alpha x(t)$ is given by $x(t) = x(0)\exp(\alpha t)$ for $\alpha \neq 0$.

HINT: To find the characteristic curves with starting point $0 < x_0 < 1$ and to find the solution $u(x,t)$ with $x > 0$ and $1 + t > x$, you will need to carefully analyze what happens at $x = 1$.

Solution: The characteristics equation for the equation (4.1) are,

via: du/dt + du/dx dx/dt = Du/Dt
=> dx/dt = a(x)

$$\frac{dx(t)}{dt} = a(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } 0 < x < 1, \\ 1 & \text{if } 1 \leq x. \end{cases} \quad \text{with} \quad x(0) = x_0 \in \mathbb{R}.$$

Integrating all three cases, we obtain

$$x(t) = \begin{cases} x_0 & \text{if } x_0 \leq 0, \\ \min(x_0 \exp(t), 1 + \log x_0 + t) & \text{if } 0 < x_0 < 1, \\ x_0 + t & \text{if } 1 \leq x_0. \end{cases}$$

x=x0*exp(t)
=> x0=x*exp(-t)

Since the solution is constant along characteristic curves $x(t)$, it is given by

x0=exp(x-1-t) &&
x0<1 => exp(x-1-t)<1
=> x < (1+t)

$$u(x,t) = \begin{cases} u_0(x) & \text{if } x \leq 0, \\ u_0(x\exp(-t)) & \text{if } 0 < x < 1 \\ u_0(\exp(x - t - 1)) & \text{if } 1 \leq x < 1 + t, \\ u_0(x - t) & \text{if } x > 0 \text{ and } 1 + t \leq x. \end{cases}$$

Vorgehen FVM (Characteristics, Burgers, Shock&Rarefaction)

Take equation and compare to $\frac{\partial u}{\partial t}\frac{\partial t}{\partial s} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} = \frac{du}{ds}$. If rhs is 0, the solution is constant on the characteristic. Parametrize, usually $x = x(s), t = t(s) \Rightarrow \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} = \frac{du}{dt}$ in an easier version. Then figure out $\frac{\partial x}{\partial t}$ is where x is the characteristic (For Burgers, the Characteristic is simply $x(t) = x_0 + u_0 * t$). Take that and reformulate to $x_0 = \cdots$, then insert into $u_0(x_0)$ to get $u(x)$. Sketch the characteristics, set them equal to find where they intersect (time and space) and build Shock/Rarefaction as described below. if $U_L > U_R$, it's a shock. Use the characteristic as condition and the $u_0(x_0)$ as value. only consider t until the relation between left and right would switch. if that happens, solve that new problem again.

Rankine-Hugoniot ($U_L > U_R$)

Shock speed $s(t) = \frac{f(U_R(t)) - f(U_L(t))}{U_R(t) - U_L(t)}$ for a weak solution U, with U_L left of the shock and U_R right of the shock. f is the flux (usually multiplied with $\frac{\partial}{\partial x}$)

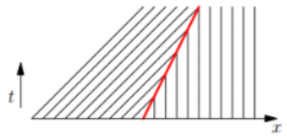
this is only the speed of the shock, so the position of the shock would be $x_0 + s \cdot t$.

This gives for the boundary terms regarding x a value dependent on t - that mus fullfill the initial data, so choose x_0 correspondingly.

e.g. $u(x,t) = \begin{cases} U_L, & if\ x \leq -t \\ U_R, & if\ x > -t \end{cases}$

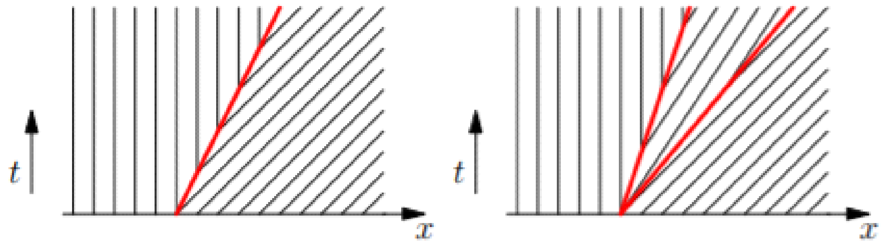
Lax-Entropy condition

Requirement that information is not generated by the shock but comes from the initial data instead. Characteristics flow from the x-Axis into the shock, not out of it. This Condition is TVD - that is, the BV norm decreases in time.



good

$U_L = 1, U_R = 0, x = 0.5t$



bad (create rarefaction instead)

$U_L = 0, U_R = 1, x = 0.5t \; bzw \; x = \frac{1}{3} \& \frac{5}{6}$

$U_L(t) > s(t) > U_R(t),$ $\quad$ $for\ burgers$

$f'(U_L(t)) > s(t) > f'(U_R(t)),$ $\quad$ $for\ general\ convex\ f$

Entropy solutions are TVD

Rarefaction waves ($U_L < U_R$)

Assuming self-similarity: the solutions only depend on the ratio x/t

f is assumed in proof to be strictly convex, so f' is strictly increasing.

Create rarefaction solution as $V\left(\frac{x}{t}\right) = (f')^{-1}\left(\frac{x}{t}\right)$ assuming $V_\xi \neq 0$ in $\left(f'\left(V(\xi)\right) - \frac{x}{t}\right) = 0$.

$$\Rightarrow U(x,t) = \begin{cases} U_l, & if\ x \le f'(U_l)\boldsymbol{t} \\ (f')^{-1}\left(\frac{x}{t}\right), & if\ f'(U_l)t < x \le f'(U_r)t \\ U_r, & if\ x > f'(U_r)\boldsymbol{t} \end{cases}$$

remember to multiply with t -----------------------^

Weak Solution

(3.4) $\quad U_t + f(U)_x = 0$

$$(3.15) \qquad \int_{\mathbb{R}\times\mathbb{R}_+} U\varphi_t + f(U)\varphi_x \, dxdt + \int_{\mathbb{R}} U_0(x)\varphi(x,0)\, dx = 0.$$

Definition 3.2 (Weak solution). *A function $U \in L^\infty(\mathbb{R}\times\mathbb{R}_+)$ is a* weak solution *of (3.4) with initial data $U_0 \in L^\infty(\mathbb{R})$ if the identity (3.15) holds for all test functions $\varphi \in C_c^1(\mathbb{R}\times\mathbb{R}_+)$.*

Note that the identity (3.15) is well-defined as long as $U \in L^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}_+)$.

If a Weak solution U is also differentiable, then U satisfies (3.4) point-wise. Hence, the class of weak solutions contains classical solutions.

Weak solutions are not neccessarily differentiable or continuous.

Oleinik entropy condition

If the flux f is not strictly convex.

$$s(t) \le \frac{f(k) - f(U_L)}{k - U_L}, for\ all\ k. U_L \le k \le U_R$$

Finite Volume Method

Domain $[x_L, x_R]$. Point discretisation $x_j = x_L + \left(j + \frac{1}{2}\right)\Delta x\ ,\ j = 0, \dots, N$ where $\Delta x = \frac{x_R - x_L}{N+1}$.

Midpoint values are in between: $x_{j-\frac{1}{2}} = x_j - \frac{\Delta x}{2} = x_L + j\Delta x$

These midpoints are the borders of the cells. So the middle of the cell is still with an integer index.

Point value approximation does not work because there are points where the solution is not continuous. So we use cell averages $U_j^n \approx \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} U(x, t_n)\ dx$

<u>Conservation Law</u> $U_t + f(U)_x = 0$

$$Integrated: \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} U(x, t_{n+1})\, dx - \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} U(x, t_n) = -\int_{t_n}^{t_{n+1}} f\left(U\left(x_{j+\frac{1}{2}}, t\right)\right)\, dx + \int_{t_n}^{t_{n+1}} f\left(U\left(x_{j-\frac{1}{2}}, t\right)\right)\, dt$$

This gives $F^n_{j+\frac{1}{2}} = \frac{1}{\Delta t}\int_{t_n}^{t^{n+1}} f\left(U\left(x_{j+\frac{1}{2}}, t\right)\right) dt$

and $U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(F^n_{j+\frac{1}{2}} - F^n_{j-\frac{1}{2}}\right)$

This is **not explicit** because F needs knowledge of the exact solution U.

Godunov

Riemann Problem at each cell interface $\begin{cases} U_t + f(U)_x = 0 \\ U(x,t^n) = \begin{cases} U_j^n, \ if\ x < x_{j+\frac{1}{2}} \\ U_{j+1}^n, \ if\ x > x_{j+\frac{1}{2}} \end{cases} \end{cases}$

Solutions of this Riemann Problem are self-similar

$\overline{U}_j(x,t) = \overline{U}_j\left(\frac{x - x_{j+\frac{1}{2}}}{t - t^n}\right)$

Wavespeed is bounded by $\max_j \left|f'\left(U_j^n\right)\right|$ so **CFL**: $\max_j \left|f'\left(U_j^n\right)\right| \frac{\Delta t}{\Delta x} \le \frac{1}{2}$
to ensure that the neighboring problems do not interact before the next time level.
Solution is constant when ξ is.

Explicit Godunov Flux:

$F^n_{j+\frac{1}{2}} = f\left(\overline{U}_j(0+)\right) = f\left(\overline{U}_j(0-)\right)$ at the edge

$F^n_{j+\frac{1}{2}} \approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f\left(U\left(x_{j+\frac{1}{2}}, t\right)\right) dt$

Finite Volume Scheme: $U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(F^n_{j+\frac{1}{2}} - F^n_{j-\frac{1}{2}}\right)$

Godunov Flux: $F^n_{j+\frac{1}{2}} = F\left(U_j^n, U_{j+1}^n\right) =$
$\begin{cases} \min_{U_j^n \le \theta \le U_{j+1}^n} f(\theta), \ if\ U_j^n \le U_{j+1}^n \\ \max_{U_{j+1}^n \le \theta U_j^n} f(\theta), \ if\ U_{j+1}^n \le U_j^n \end{cases}$ also valid for non-convex flux

These two together are the Godunov Scheme

Computing it: If flux f has a single minimum ω and no local maxima it is strictly convex, then

$F^n_{j+\frac{1}{2}} = F\left(U_j^n, U_{j+1}^n\right)$
$= \max\left(f\left(\max\left(U_j^n, \omega\right)\right), f\left(\min\left(U_{j+1}^n, \omega\right)\right)\right)$

(local means without global and only in the interval of the starting values)

Note: dt = CFL * dx / max{1,max |f'|} in der Übung

"The point is that the derivative of the BL-flux is increasing around the initial values we gave them, therefore, the normal CFL condition won't really work since f'(u) is not bounded by the values in the cells initially. A smaller CFL number does the trick (around 0.2), and the max of f'(u) and 1 also works (you could just do the global max of f'(u)). "

Lemma for two-point flux FVM

the scheme is monotone if F is Lipschitz continuous, diffable in both arguments, non-decreasing in first argument and non-increasing in second argument. This involves $\left|\frac{\partial F}{\partial x_1}(b,c)\right| + \left|\frac{\partial F}{\partial x_2}(a,b)\right| \le \frac{\Delta x}{\Delta t}$

(Linearized) Roe

Approximates the exact solution that Godunov would needs.

$$F^n_{j+\frac{1}{2}} = F^{Roe}\left(U^n_j, U^n_{j+1}\right) = \begin{cases} f\left(U^n_j\right), & if\ \hat{A}_{j+\frac{1}{2}} \geq 0 \\ f\left(U^n_{j+1}\right), & if\ \hat{A}_{j+\frac{1}{2}} < 0 \end{cases}$$

$$\hat{A}_{j+\frac{1}{2}} = \begin{cases} \dfrac{f\left(U^n_{j+1}\right) - f\left(U^n_j\right)}{U^n_{j+1} - U^n_{j+1}}, & if\ U^n_{j+1} \neq U^n_j \\ f'\left(U^n_j\right), & if\ U^n_{j+1} = U^n_j \end{cases}$$

$$f(U)_x = f'(U)U_x \approx \hat{A}_{j+\frac{1}{2}} U_x$$

$\hat{A}$ is assumed constant between two cells.
Use with the finite volume Scheme above.
Is simpler to implement than Godunov, because no optimization problem. Can be as good as Godunov in 1\0 case, but can also fail af in 0/1 case. It fails at solving (some) rarefactions because a rarefaction wave can travel in both directions but the approximated $\hat{A}$ is linearized.

Lax-Friedrichs scheme

Two waves from the middle to the left and the right.

$$s^l_{j+\frac{1}{2}} = -\frac{\Delta x}{\Delta t}, \qquad s^r_{j+\frac{1}{2}} = \frac{\Delta x}{\Delta \text{t}}$$

$$F^n_{j+\frac{1}{2}} = F^{LxF}\left(U^n_j, U^n_{j+1}\right) = \frac{f\left(U^n_j\right) + f\left(U^n_{j+1}\right)}{2} - \frac{\Delta x}{2\Delta t}\left(U^n_{j+1} - U^n_j\right)$$

stable, nonoscillatory, approx. the entropy solution, but computed solutions are diffusive. Shocks are smeared.
Results inferior to Godunovs.

Rusanov Scheme

Instead of the maximum speed, use locally selected speeds $s^r_{j+\frac{1}{2}} = -s^l_{j+\frac{1}{2}} = s_{j+\frac{1}{2}} = \max\left(\left|f'\left(U^n_j\right)\right|, \left|f'\left(U^n_{j+1}\right)\right|\right)$

<u>Rusanov Flux</u>: $F^n_{j+\frac{1}{2}} = F^{Rus}\left(U^n_j, U^n_{j+1}\right) = \frac{f\left(U^n_j\right)+f\left(U^n_{j+1}\right)}{2} - \frac{\max\left(\left|f'\left(U^n_j\right)\right|,\left|f'\left(U^n_{j+1}\right)\right|\right)}{2}\left(U^n_{j+1} - U^n_j\right)$

Comparison FEM, FDM, FVM (TODO)

https://math.stackexchange.com/questions/424672/what-is-the-difference-between-finite-difference-methods-finite-element-methods#1359419
<https://scicomp.stackexchange.com/questions/290/what-are-criteria-to-choose-between-finite-differences-and-finite-elements>

FDM

efficient quadrature-free implementation
local conservation for certain schemes
robust nonlinear methods for transport
discrete maximum principle for some problems
b u t
no Galerkin orthogonality, so convergence may be difficult to prove
Boundary conditions tend to be complicated to implement
Stencil grows if physics includes "cross terms"
//wtf did I just write?

FEM requires f to be non-zero, continous and integrable

conservation law may be violated (e.g. with shocks)
higher order accuracy is achieved by using higher order basis shape functions
Suitable for Heat transfer, Structural mechanics, vibrational analysis. Generally ideal for linear PDEs, but expensive and complex for non-linear PDEs.
FEM have the benefit of being very flexible, e.g., the grids may be very non-uniform and the domains may have arbitrary shape.
Galerkin orthogonality (discrete solution to coercive problems is within a constant of the best solution in the space)
simple geometric flexibility
robust transport algorithm

cellwise entropy inequality guarantees L^2 stability holds independent of mesh, dimension, order of accuracy and presence of discontinuous solutions, without needing nonlinear limiters
easy to implement boundary conditions
"can choose conservation statement by choosing test space"
high order accuracy even with discontinuous coefficients, as long as you can align to boundaries
Continuous fem has trouble with transport (diffusive and oscillatory)
Have to choose between consistent mass matrix (some nice properties but has full inverse, thus requiring an implicit solve per time step) and lumped mass matrix
Has usually more nonzeros in assembled matrices
Variational Method: e.g. energies always drop for certain equations
Nice for irregular meshes

FVM is over volume. Fluxes arre integrated and flux is conserved. Can handly almost any PDEs. Accuracy is based on order of polynomial used for interpolation (?). Ideal for Fluid mechanics.

Bilinear Form

Definition 5.2.2 (Bilinear form): Let $a : V \times V \to \mathbb{R}$ be such that

$$a(\alpha v + \beta \tilde{v}, w) = \alpha \cdot a(v, w) + \beta \cdot a(\tilde{v}, w) \qquad (5.23)$$
$$a(v, \alpha w + \beta \tilde{w}) = \alpha \cdot a(v, w) + \beta \cdot a(v, \tilde{w}) \qquad (5.24)$$

$\forall v, \tilde{v}, w, \tilde{w} \in V$. Then, a is defined as a bilinear form.

Example 5.2.5: $(\cdot,\cdot)$ in a Hilbert space is a "symmetric" bilinear form.

Definition 5.2.3 (Linear form): Let $L : V \to \mathbb{R}$ be such that:

$$L(\alpha v + \beta w) = \alpha L(v) + \beta L(w)\,. \qquad \forall v, w \in V \qquad (5.25)$$

Then, L is a linear form

Example 5.2.6: Fix $f \in L^2(\Omega)$, then

$$L_f(g) = \int_\Omega f g \, \mathrm{d}x \qquad (5.26)$$

defines a linear form.

(1b) Specify the bilinear form and linear form in the variational formulation obtained in the previous subtask.

Solution:

$$\mathsf{a}(u, v) = \int_\Omega (x^2 + y^2 + 1) \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}, \quad l(v) = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}.$$

Convex

if second derivative > 0

(5a) Most of the examples in the lecture slides are for **convex** flux functions! Determine the following:

- Is the flux function $f(u) = (u^3)/3$ **convex** for all values of $u \in \mathbb{R}$?
- Is the flux function $f(u) = (u^3)/3$ **convex** for all values of $u \in [0, 1]$?

Is the convexity of f in the interval $[0, 1]$ sufficient if we consider only initial data given in (5.2)? Explain why.

Solution: The flux function is convex only in the interval $[0, \infty)$. Since the scalar conservation laws satisfy the minimum/maximum principles, the support of the solution u is contained in the interval $[\min u_0, \max u_0]$ at all times $t > 0$.

Lax-Milgram Theorem: unique solution

If B is a sesquilinearform (or bilinear), and coercive on H ($B(x, x) \geq c\,||x||^2$ for some $c > 0$)

then given a $w \in H$, there exists a unique element $x \in H$ such that $B(u,x) = \langle u, w\rangle$ for all $u \in H$. For such x one has $||x|| \leq \frac{1}{c}||w||$ where $c > 0$ is the bound from below of the form (That is, $B(u,u) \geq c||u||^2\ \forall u \in H$)

Let V be a reflexive Banach space (cf. also Reflexive space) and let $b : V \times V \longrightarrow \mathbb{C}$ be a sesquilinear mapping (bilinear when b is real; cf. also Sesquilinear form) such that

$$|b(u,v)| \leq M\|u\|.\|v\|,\ u, v \in V$$

(continuity) and

$$|b(u,u)| \geq \gamma\,\|u\|^2,\ u \in V$$

(strong coercivity), where $M, \gamma > 0$. Then there exists a unique bijective linear mapping $B: V \to V'$, continuous in both directions and uniquely determined by b, with

$$b(u,v) = \overline{\langle Bu, v\rangle}, \qquad \forall u, v \in V,$$

$$b(B^{-1}l, v) = \overline{\langle l, v\rangle}, \qquad \forall v \in V, l \in V',$$

and for the norms one has:

$$\|B\|_{\mathcal{L}(V,V')} \leq M,$$

$$\|B^{-1}\|_{\mathcal{L}(V',V)} \leq \frac{1}{\gamma}.$$

This implies that $u = B^{-1}l$ is the solution of (a1). The above theorem only establishes existence of a solution to (a1), namely $u = B^{-1}l$, but does not say anything about the construction of this solution. In 1965, W.V. Petryshyn

Let ϕ be a bounded coercive bilinear form on a Hilbert space H. The Lax-Milgram theorem states that, for every bounded linear functional f on H, there exists a unique $x_f \in H$ such that

$$f(x) = \phi(x, x_f) \tag{1}$$

for all $x \in H$.

Example

Notice: Coercivity is with a(v,v)

(1b) Suppose now for this subtask that $h = 0$. Show that the solution to the variational formulation in (1a) exists and it is unique.

HINT: Use Lax-Milgram Lemma.

Solution: Since u does not satisfy homogeneous Dirichlet boundary conditions on Γ_D, before applying Lax-Milgram Lemma we use the offset function technique. Indeeed, the variational formulation can be restated as:

$$\text{Find } u = u_0 + u_g,\ u_0 \in H^1_{0,\Gamma_D}(\Omega),\ u_g|_{\Gamma_D} = g, \text{ such that}$$
$$a(u_0, v) = l(v) - a(u_g, v) := \tilde{l}(v) \quad \text{for all } v \in H^1_{0,\Gamma_D}(\Omega),$$

with bilinear and linear form defined as in (1.4).

We have to prove that the hypothesis of the Lax-Milgram Lemma are fulfilled considering $\tilde{l}$ instead of l as right-hand side:

- continuity of the bilinear form:

  $$|a(u_0, v)| \leq |u_0|_{H^1(\Omega)}|v|_{H^1(\Omega)} + C_1\|u_0\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \leq (1 + C_1)\|u_0\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)},$$

  where $C_1 := \sup_{\boldsymbol{x}\in\Omega} c(\boldsymbol{x})$ and we used the Cauchy-Schwarz inequality. Alternatively, one can use Poincaré's inequality.

- coercivity of the bilinear form:

  $$a(v, v) \geq |v|^2_{H^1(\Omega)} + C_2\|v\|^2_{L^2(\Omega)} \geq \min\{1, C_2\}\|v\|^2_{H^1(\Omega)},$$

  with $C_2 := \inf_{\boldsymbol{x}\in\Omega} c(\boldsymbol{x}) > 0$. Alternatively, one can simply observe that $|v|^2_{H^1(\Omega)} + C_2\|v\|^2_{L^2(\Omega)} \geq |v|^2_{H^1(\Omega)}$, which is sufficient thanks to Poincaré's inequality.

- continuity of the linear form:
  we have

  $$|l(v)| \leq \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)}$$

  (where we used the Cauchy-Schwarz inequality); thus, thanks to the continuity of $a(\cdot,\cdot)$:

  $$|\tilde{l}(v)| \leq \|f\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)} + (1 + C_1)\|u_g\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} \leq C_3\|v\|_{H^1(\Omega)}$$

  with $C_3 := \|f\|_{L^2(\Omega)} + (1 + C_1)\|u_g\|_{H^1(\Omega)}$, which is bounded because g continuous on the boundary implies that $u_g \in H^1(\Omega)$.

Solution: We check that the assumptions of the Lax-Milgram Lemma are fulfilled:

- Continuity of the bilinear form:
  using the Cauchy-Schwarz inequality we get

$$|\mathsf{a}(u,v)| \leq \sup_{\boldsymbol{x}\in\Omega}(x^2+y^2+1)|u|_{H^1(\Omega)}|v|_{H^1(\Omega)} \leq 2|u|_{H^1(\Omega)}|v|_{H^1(\Omega)} \tag{1.5}$$

  for every $u, v \in H^1_{0,\Gamma_D}(\Omega)$.

- Coercivity of the bilinear form:

$$\mathsf{a}(v,v) \geq \inf_{\boldsymbol{x}\in\Omega}(x^2+y^2+1)|v|^2_{H^1(\Omega)} \geq |v|^2_{H^1(\Omega)} \tag{1.6}$$

  for every $v \in H^1_{0,\Gamma_D}(\Omega)$.

- Continuity of the linear form:

$$|l(v)| \leq \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \leq C_P\|f\|_{L^2(\Omega)}|v|_{H^1(\Omega)} \tag{1.7}$$

  for every $v \in H^1_{0,\Gamma_D}(\Omega)$, where we have used the Cauchy-Schwarz and the Poincaré inequalities (with C_P the Poincaré constant).

Find $u \in V$ such that:

$$a(u,v) = L(v) \qquad \forall v \in V \tag{5.27}$$

In general Eqn 5.27 may not be solvable, however under certain assumptions on a and L, we have the following result:

Theorem 5.2.1: Let the bilinear form a have the following properties:

1. a is symmetric, i.e.

$$a(u,v) = a(v,u) \qquad \forall u,v \in V$$

2. a is continuous, i.e.

$$|a(u,v)| \leq r\|u\|_V\|v\|_V \qquad \text{for } r > 0$$

3. a is coercive, i.e. $\exists \alpha > 0$ such that

$$|a(v,v,)| \geq \alpha\|v\|^2_V$$

   And let the linear form L satisfy

4. L is continuous, i.e.

$$|L(v)| \leq \Lambda\|v\|_V \qquad v \in V$$

Then, the variational problem (Eqn 5.27) has a unique solution $u \in V$. Furthermore, the solution satisfies the stability estimate:

$$\|u\|_V \leq \frac{\Lambda}{\alpha}. \tag{5.28}$$

Given the functional J as

$$J: \begin{array}{ccc} V & \longrightarrow & \mathbb{R} \\ \Psi & & \Psi \\ v & \longmapsto & J(v) = \frac{1}{2}a(v,v) - L(v) \end{array} \tag{5.29}$$

then the solution u of Eqn 5.27 satisfies

$$J(u) \leq \inf_{v\in V} J(v) \tag{5.30}$$

Remark 5.2.1: The existence of a solution u of Eqn 5.27 is a consequence of the Lax-Milgram lemma.

for the upper bound (5.28), Λ and α come from coercivity and continuity above.

Poincaré inequality (for coerciveness)

Let p so that $1 \leq p < \infty$ and Ω a subset with at least one bound. Then there exists a constant C, depending only on Ω and p so that for every function u of the $W_0^{1,p}(\Omega)$ Sobolev , $\left\|u - u_\Omega\right\|_{L^p(\Omega)} \leq C\,\left\|\nabla u\right\|_{L^p(\Omega)},\ \ where\ \ u_\Omega =$

$\frac{1}{|\Omega|}\int_\Omega u(y)\,dy$ *is the average value of u over* Ω

$$e.g.\quad \int_0^1 vf\,dx \le ||v||_{L^2}\,||f||_{L^2} \le ||v||_{H_0^1}\,||f||_{L^2}\,, \qquad (4.12)$$

also works for H^1 instead of H_0^1.

EMP: $||u||_{L^p(\Omega)} \le c||\nabla u||_{L^p(\Omega)} \quad 1 \le p < \infty$
$\Rightarrow ||v||_{L^2} \le C||v||_{H^1}$

Cauchy-Schwarz
$|\langle u,v\rangle|^2 \le |\langle u,u\rangle| \cdot |\langle v,v\rangle| = ||u|| \cdot ||v||$

Notice: Nabla disappears (stammfnkt nehmen)
For $u \in H_0^1$, $\int_0^1 u'v'dx \ \le ||u||_{H_0^1}\,||f||_{H_0^1}$
$\int u\,v\,dx \le ||u||_{L^2(\Omega)}||v||_{L^2(\Omega)}$
With an additional factor, we just multiply by its supremum or infinimum instead to prove ≤ or ≥
$\left|\int (x^2+y^2+1)\nabla \mathrm{u}(\mathrm{x}) \cdot \nabla \mathrm{v}(\mathrm{x})\,dx\right| \le \sup(x^2+y^2+1)\,|u|_{H^1(\Omega)}|v|_{H^1(\Omega)} \le 2|u|_{H^1}|v|_{H^1}$ *in this case*

Loglog plot
slope = polynomial exponent
smooth solution of a variational problem in one excercise had convergence 2 in L2 and 1 in H1

Monotonicity
Entropy solutions are monotonicity preserving. i.e. if all initial values of one are smaller than the other, then this holds at every timestep.

LF, Rusanov, Godunov and Engquist are monotonic schemes. They are also consistent and conservative.
Consistent Monotone schemes are TVD

Example Maximum Principle
use $x_i = i\Delta x$ and make sure summands in step to U_i^{n+1} before the U^n are ≥ 0. Then we can say that $\max(aU_i^{n+1}) = \max(U_i^{n+1})$ · a (in multiple summands) and solve to get $\le \max(U_i^0)$ *from it*

Exploting the fact that

$$r_i = i\Delta r \quad \text{and} \quad r_{i\pm\frac{1}{2}} = (i \pm \frac{1}{2})\Delta r,$$

we get that

$$U_i^{n+1} = U_i^n + \frac{\Delta t}{i\Delta r^2}\left((i+\frac{1}{2})\left(U_{i+1}^n - U_i^n\right) - (i-\frac{1}{2})\left(U_i^n - U_{i-1}^n\right)\right), \qquad n = 1,\dots,M,\ i = 1,\dots,N. \tag{17}$$

2e)

Recall that we say the scheme obeys the maximum condition if

$$\max_i U_i^n \le \max_i U_i^0 \qquad n = 1, 2, \dots.$$

Find necessary conditions on Δt for the maximum condition to be fulfilled for the scheme derived in the previous exercise.

Solution: We note that the scheme is equivalent to

$$U_i^{n+1} = \left(1 - \frac{\Delta t}{r_i\Delta r^2}\left(r_{i+\frac{1}{2}} + r_{i-\frac{1}{2}}\right)\right)U_i^n + \frac{\Delta t}{r_i\Delta r^2}r_{i+\frac{1}{2}}U_{i+1}^n + \frac{\Delta t}{r_i\Delta r^2}r_{i-\frac{1}{2}}U_{i-1}^n, \qquad n = 1,\dots,M,\ i = 1,\dots,N. \tag{18}$$

In order to bound the maximum, we need to guarantee that each coefficient in front of $U^n_\cdot$ is non-negative. That is, we require that

$$1 - \frac{\Delta t}{r_i\Delta r^2}\left(r_{i+\frac{1}{2}} + r_{i-\frac{1}{2}}\right) \ge 0 \quad , \frac{\Delta t}{r_i\Delta r^2}r_{i+\frac{1}{2}} \ge 0 \quad \text{and} \quad \frac{\Delta t}{r_i\Delta r^2}r_{i-\frac{1}{2}} \ge 0.$$

The latter two inequalities are obviously fulfilled since $i \ge 0$ and $\Delta r > 0$. For the former, we must therefore require that

$$1 - \frac{\Delta t}{r_i\Delta r^2}\left(r_{i+\frac{1}{2}} + r_{i-\frac{1}{2}}\right) \ge 0$$

that is

$$\frac{\Delta t}{r_i\Delta r^2}\left(r_{i+\frac{1}{2}} + r_{i-\frac{1}{2}}\right) \le 1$$

inserting for r_i and $r_{i\pm\frac{1}{2}}$, we see that

$$\begin{aligned}\frac{\Delta t}{r_i\Delta r^2}\left(r_{i+\frac{1}{2}} + r_{i-\frac{1}{2}}\right) &= \frac{\Delta t}{i\Delta r\Delta r^2}\left((i+\frac{1}{2})\Delta r + (i-\frac{1}{2})\Delta r\right) \\ &= \frac{\Delta t}{i\Delta r\Delta r^2}2i\Delta r \\ &= \frac{2\Delta t}{\Delta r^2},\end{aligned}$$

that is

$$\Delta t \le \frac{\Delta r^2}{2}. \tag{19}$$

We then get that

$$\begin{aligned}\max_i U_i^{n+1} &= \max_i \left\{\left(1 - \frac{\Delta t}{r_i\Delta r^2}\left(r_{i+\frac{1}{2}} + r_{i-\frac{1}{2}}\right)\right)U_i^n + \frac{\Delta t}{r_i\Delta r^2}r_{i+\frac{1}{2}}U_{i+1}^n + \frac{\Delta t}{r_i\Delta r^2}r_{i-\frac{1}{2}}U_{i-1}^n\right\} \\ &\le \left(1 - \frac{\Delta t}{r_i\Delta r^2}\left(r_{i+\frac{1}{2}} + r_{i-\frac{1}{2}}\right)\right)\max_i\{U_i^n\} + \frac{\Delta t}{r_i\Delta r^2}r_{i+\frac{1}{2}}\max_i\{U_{i+1}^n\} + \frac{\Delta t}{r_i\Delta r^2}r_{i-\frac{1}{2}}\max_i\{U_{i-1}^n\} \\ &= \max\{U_i^n\},\end{aligned}$$

and the maximum principle follows.

possibly useful: **Truncation** error from Skript nearby Heat Eq

Implicit FD Convergence with second order in space and first in time

$$\left|T_j^n\right| = \left|\frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{u_{j+1}^n - 2x_j^n + n_{j+1}^n}{\Delta x^2}\right| \le C\left(\Delta t + \Delta x^2\right) \text{ (it can be shown that...)} \Rightarrow$$

$$if\ CFL\ holds\ \sqrt{\frac{\Delta x}{2}\sum_{j=1}^N \left|f_j - u_j^{n}\right|^2} \le C^2(\Delta t + \Delta x^2)$$

explicit FD

$\le C(\Delta t^2 + \Delta x^2) \Rightarrow$ Convergence with second order in space&time

Consistency, Stability

Consistency: a finite difference approximation is consistent if the **truncation error** approaches zero while we decrease mesh- and timestep-size. Probably show it with Taylor

Consider the following discretization scheme:

$$U_j^{n+1} = U_j^n - \Delta t \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2}, \quad \text{with} \quad \Delta t = h. \tag{3.10}$$

(3b) Show, that (3.10) is a consistent discretization of (3.1), i.e. that as $h \to 0$, scheme (3.10) approximates the continuous problem (3.1).

Solution: Using Taylor series, we observe that

$$u_{xx}(x_j, t^n) \approx \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2}, \quad \forall j = 2, \ldots, N-1, \tag{3.11}$$

and

$$u_t(x_j, t^n) \approx \frac{U_j^{n+1} - U_j^n}{\Delta t}, \quad \forall j = 1, \ldots, N. \tag{3.12}$$

Thus, (3.10) is obtained by inserting (3.11) and (3.12) into (3.1), and hence is a consistent approximation as $h \to 0$ and $\Delta t \to 0$.

Stability: a finite difference approximation is stable if the complete error decays with the marching step
Convergence is when stable and consistent with the PDE. (*Lax Equivalence Theorem*)

ODE

Runge-Kutta

Definition 11.4.9. Explicit Runge-Kutta method

For $b_i, a_{ij} \in \mathbb{R}$, $c_i := \sum_{j=1}^{i-1} a_{ij}$, $i, j = 1, \ldots, s$, $s \in \mathbb{N}$, an s-stage explicit Runge-Kutta single step method (RK-SSM) for the ODE $\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y})$, $\mathbf{f} : \Omega \to \mathbb{R}^d$, is defined by ($\mathbf{y}_0 \in D$)

$$\mathbf{k}_i := \mathbf{f}(t_0 + c_i h, \mathbf{y}_0 + h \sum_{j=1}^{i-1} a_{ij} \mathbf{k}_j), \quad i = 1, \ldots, s \quad , \quad \mathbf{y}_1 := \mathbf{y}_0 + h \sum_{i=1}^{s} b_i \mathbf{k}_i .$$

The vectors $\mathbf{k}_i \in \mathbb{R}^d$, $i = 1, \ldots, s$, are called increments, $h > 0$ is the size of the timestep.

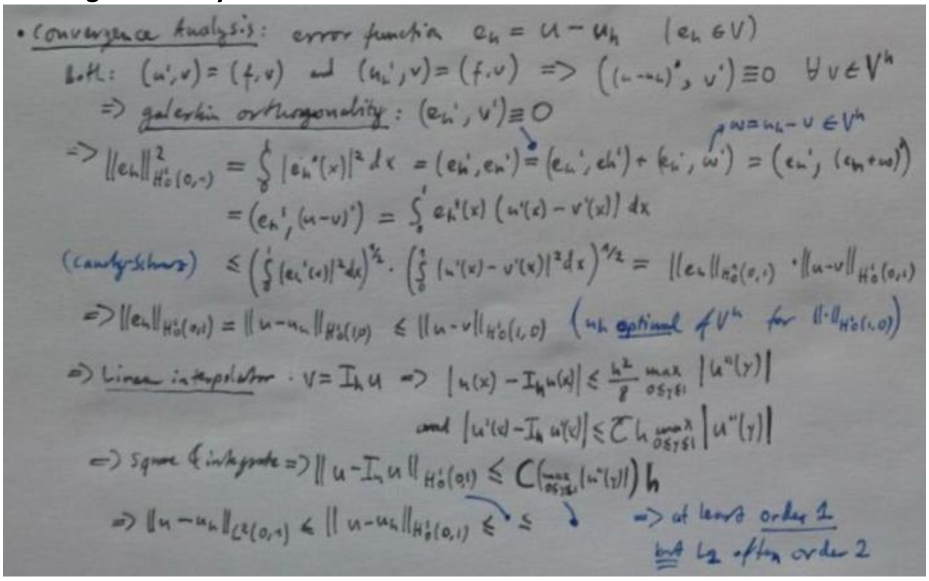
Definition 12.3.18. General Runge-Kutta single step method (*cf.* Def. 11.4.9)

For $b_i, a_{ij} \in \mathbb{R}$, $c_i := \sum_{j=1}^{s} a_{ij}$, $i, j = 1, \ldots, s$, $s \in \mathbb{N}$, an s-stage Runge-Kutta single step method (RK-SSM) for the IVP (11.1.20) is defined by

$$\mathbf{k}_i := \mathbf{f}(t_0 + c_i h, \mathbf{y}_0 + h \sum_{j=1}^{s} a_{ij} \mathbf{k}_j), \quad i = 1, \ldots, s \quad , \quad \mathbf{y}_1 := \mathbf{y}_0 + h \sum_{i=1}^{s} b_i \mathbf{k}_i .$$

As before, the $\mathbf{k}_i \in \mathbb{R}^d$ are called increments.

Convergence Analysis


• Convergence Analysis: error function $e_h = u - u_h$ $(e_h \in V)$
both: $(u', v') = (f, v)$ and $(u_h', v') = (f, v)$ ⇒ $((u - u_h)', v') \equiv 0$ $\forall v \in V^h$
⇒ galerkin orthogonality: $(e_h', v') \equiv 0$
⇒ $\|e_h\|^2_{H_0^1(0,1)} = \int_0^1 |e_h'(x)|^2 dx = (e_h', e_h') = (e_h', e_h') + (e_h', w')$ [$w = u_h - v \in V^h$] $= (e_h', (e_h + w)')$
$= (e_h', (u - v)') = \int_0^1 e_h'(x)(u'(x) - v'(x))\, dx$
(Cauchy-Schwarz) $\leq \left(\int_0^1 |e_h'(x)|^2 dx\right)^{1/2} \cdot \left(\int_0^1 |u'(x) - v'(x)|^2 dx\right)^{1/2} = \|e_h\|_{H_0^1(0,1)} \cdot \|u - v\|_{H_0^1(0,1)}$
⇒ $\|e_h\|_{H_0^1(0,1)} = \|u - u_h\|_{H_0^1(0,1)} \leq \|u - v\|_{H_0^1(0,1)}$ (u_h optimal of V^h for $\|\cdot\|_{H_0^1(0,1)}$)
⇒ Linear interpolation: $v = I_h u$ ⇒ $|u(x) - I_h u(x)| \leq \frac{h^2}{8} \max_{0 \leq y \leq 1} |u''(y)|$
and $|u'(x) - I_h u'(x)| \leq C h \max_{0 \leq y \leq 1} |u''(y)|$
⇒ square & integrate ⇒ $\|u - I_h u\|_{H_0^1(0,1)} \leq C\left(\max_{0 \leq y \leq 1} |u''(y)|\right) h$
⇒ $\|u - u_h\|_{L^2(0,1)} \leq \|u - u_h\|_{H_0^1(0,1)} \leq$ ⇒ at least order 1
but L_2 often order 2

Separation der Variablen

$y' = f(x)g(y), \dfrac{dy}{dx} = f(x)g(y)$

1. Nullstellen: $g(y_0) = 0 \Rightarrow y(x) = y_0$

2. separieren: $\frac{dy}{dx} = f(x)g(y) \Rightarrow \frac{1}{g(y)} dy = f(x)dx$

3. integrieren: $\int_{y_0}^{y} \frac{1}{g(\tilde{y})} d\tilde{y} = \int_{x_0}^{x} f(\tilde{x}) d\tilde{x} + C$

4. nach y auflösen: prüfen und einsetzen

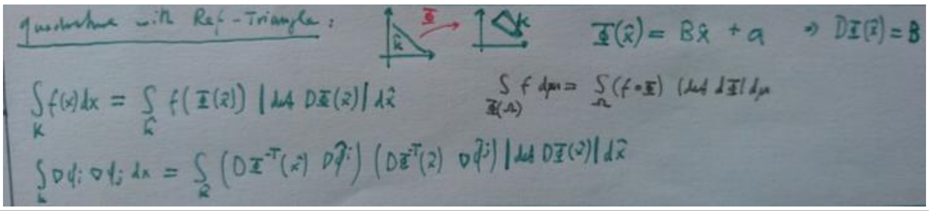
Transformation

(1g) Consider now a generic triangle K with vertices $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$, considered as *column vectors* and numbered counterclockwise. The element matrix A_K for the triangle K is given by $A_K = A_{K,1} + A_{K,2}$, with the part $A_{K,1}$ associated to the diffusion term and the part $A_{K,2}$ associated to the reaction term. Suppose again a constant coefficient $c = 1$.
Using the shape functions $\left\{\hat{\lambda}_j\right\}_{j=1,2,3}$ and their gradients $\left\{\hat{\nabla}\hat{\lambda}_j\right\}_{j=1,2,3}$ *on the reference element*, give an expression for a generic entry $A_{K,1}(i,j)$ of $A_{K,1}$ and a generic entry $A_{K,2}(i,j)$ of $A_{K,2}$ $(i,j = 1,2,3)$.

Solution: We have $A_{K,1}(i,j) = \int_{\hat{K}} J\Phi^{-T}\hat{\nabla}\hat{\lambda}_j(\hat{\boldsymbol{x}}) \cdot J\Phi^{-T}\hat{\nabla}\hat{\lambda}_i(\hat{\boldsymbol{x}}) \det J\Phi \, \mathrm{d}\hat{\boldsymbol{x}}$, with

$$J\Phi = \begin{pmatrix} \boldsymbol{a}_2 - \boldsymbol{a}_1 \\ \boldsymbol{a}_3 - \boldsymbol{a}_1 \end{pmatrix},$$

and $A_{K,2} = \int_{\hat{K}} \hat{\lambda}_j(\hat{\boldsymbol{x}}) \cdot \hat{\lambda}_i(\hat{\boldsymbol{x}}) \det J\Phi \, \mathrm{d}\hat{\boldsymbol{x}}$.



Load Vector $L_j = \int_a^b v * h \Rightarrow \int \lambda_j * h \Rightarrow$ Auf Strecke 12 ist part bei 2 null für j=1 Kronecker Delta um alle Punkte abzugehen (Aufteilung in integral von 1 nach 2 und von 2 nach 3)

(1h) Consider $f = 0$ and a generic $h \neq 0$. Write the element load vector L_K (i.e. the vector associated to the linear form when restricted to an element) for a generic triangle K with vertices $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$, considered as *column vectors* and numbered counterclockwise.
Use the one-dimensional trapezoidal quadrature rule to compute the boundary integrals; express the element load vector in terms of the vertices $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ and function values of h.

HINT: You may use the notation

$$\delta_{ij} = \begin{cases} 1 & \text{if } e_{ij} \in \Gamma_N \\ 0 & \text{otherwise,} \end{cases}$$

with e_{ij} denoting the edge connecting the vertices i and j, for $i,j = 1,2,3$.

Solution: We have

$$L_K(1) = \delta_{12} h(\boldsymbol{a}_1)\frac{|\boldsymbol{a}_2 - \boldsymbol{a}_1|}{2} + \delta_{31} h(\boldsymbol{a}_1)\frac{|\boldsymbol{a}_1 - \boldsymbol{a}_3|}{2}$$
$$L_K(2) = \delta_{12} h(\boldsymbol{a}_2)\frac{|\boldsymbol{a}_2 - \boldsymbol{a}_1|}{2} + \delta_{23} h(\boldsymbol{a}_2)\frac{|\boldsymbol{a}_3 - \boldsymbol{a}_2|}{2}$$
$$L_K(3) = \delta_{23} h(\boldsymbol{a}_3)\frac{|\boldsymbol{a}_3 - \boldsymbol{a}_2|}{2} + \delta_{31} h(\boldsymbol{a}_3)\frac{|\boldsymbol{a}_1 - \boldsymbol{a}_3|}{2}.$$